

THE CASIMIR EFFECT ON LAAKSO SPACES

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ABSTRACT. We explore the properties of an analog to the Casimir effect on Laakso spaces such as the dependence on the separation of the plates and boundary effects. We also mention some results on the influence of complex poles in the spectral zeta function over finite approximations to Laakso spaces.

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1. INTRODUCTION

With recent advances in fractal analysis there has become available a large amount of information concerning Laplacians and their spectra over fractal spaces [10, 19, 4]. There has been progress in using this information to construct and analyze analogs to physical systems, e.g. the behavior of a photon in a fractal [2] and other “physical” consequences [1, 6, 7, 20]. The physical consequence of fractal geometry that we explore in this paper is the Casimir effect [16, 13].

In these works, the underlying space is typically a finitely ramified fractal with a symmetry condition; however, the spectrum of the Laplacian for these objects is generally not known exactly or only described as a scaled Julia set, which means the growth estimates for the eigenvalue counting functions must be used instead. Laakso spaces, whose exact spectrum the authors previously computed, enable us to avoid this complication. In [9] the authors also computed the exact eigenfunctions of the Hamiltonian with a square well potential, the spectral zeta function for certain defining sequences $\{j_n\}$, and a Casimir effect on a 1 dimensional arrangement. This paper continues the analysis of the Casimir effect on Laakso spaces.

We begin with defining Laakso spaces in a convenient manner in Section 2 where we will also give an explicit description of the spectrum of a natural Laplacian on Laakso spaces. Following this is Section 3 where we discuss the general properties and calculations for spectral zeta functions over Laakso spaces. In Section 5 we

observe that the complex dimensions appear in a model which has only finite complexity and is in principle constructible as a physical object. In Section 6 we revisit the authors' earlier work in [9] and determine the strength of the Casimir effect in a Laakso space as a function of both the defining sequence $\{j_i\}$ and the distance between the plates. Lastly, in Section 6 we construct a 3+ dimensional arrangement involving a Laakso space and show that the Casimir pressure is proportional to the inverse fourth power of the separation distance. This power has the same exponent as the classical Casimir effect between two parallel uncharged conducting plates; however, what makes this result unusual is that this exponent is not equal to the spectral dimension, d_s , plus an integer as commonly seen on fractal domains. This reflects the fact that Laplacians on Laakso spaces are truly 1-dimensional rather than d_s -dimensional operators. The rapid growth in the eigenvalue counting function is due more to the geometry of Laakso spaces and their graph approximations than the nature of the Laplacian.

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2. LAAKSO SPACES

These spaces were introduced in [11] and the spectral theory on them developed in [14, 9, 18]. We will use the construction indicated in [3] and spelled out in detail in [18]. Let $\{j_i\}$ a sequence of integers such that

$$(2.1) \quad \lim_{n \rightarrow \infty} \left(\prod_{i=1}^n j_i \right)^{1/n} = r.$$

The following construction can proceed without this restriction for any integer sequence but the limits that will be taken may not exist otherwise. Define

$$(2.2) \quad d_n = \prod_{i=1}^n j_i \quad L_n = \left\{ \frac{m}{d_n} : m = 1, \dots, d_n - 1 \right\}.$$

Where L_n will be the locations of “wormholes” of level n or lower and $L_n \setminus L_{n-1}$ the locations of the new “wormholes” at level n . We will write $I = [0, 1]$ and K for a Cantor set.

Set $F_0 = I$, $G = \{0, 1\}$, and $B_n = L_n \setminus L_{n-1} \subset I$. Let $\phi_{1,0} : F_0 \times G \rightarrow F_0$ be the projection onto the interval F_0 . Define $F_1 = F_0 \times G / \phi_{1,0}^{-1}(B_1)$. Inductively construct $\phi_{n,n-1}$ and F_n . Notice that there are also naturally defined projections $\phi_{n,m} : F_n \rightarrow F_m$. Let μ_n be the probability measure on F_n that is inherited from Lebesgue measure on F_0 .

Proposition 2.1. *The system $(F_n, \phi_{n,n-1}, \mu_n)$ is a projective system of measure spaces.*

It is important to note that F_n is a metric graph as it is a collection of line segments of all of length d_n^{-1} at nodes whose locations have a coordinate in I taken from L_n . It is this particularly regular structure for the F_n that we will use in the absence of any strict geometric self-similarity.

Definition 2.1. *The projective limit of F_n is a Laakso space with data $\{j_i\}$. This is written as $\lim_{\leftarrow} F_n = L$. There are also associated projections $\Phi_n : L \rightarrow F_n$ such that $\phi_{n,m} \circ \Phi_n = \Phi_m$ for all $m \leq n$.*

For more on projective limits of measure spaces see [8].

Theorem 2.1. *For any choice of $\{j_i\}$ such that r exists the corresponding Laakso space L is a complete geodesic metric measure space with Hausdorff dimension $1 + \frac{\log(r)}{\log(2)}$.*

That the presented construction gives a Laakso space is Lemma 4.6.1 in [17]. The properties of the Laakso spaces are proved in [11].

Let Δ_n be the self-adjoint Laplacian on F_n acting as $-\frac{d^2}{dx_e^2}$ where x_e is a coordinate on each line segment. Then the domain of Δ_n is taken to be the closure of all continuous functions on F_n that are twice differentiable when restricted to each interval and satisfy Kirchoff matching conditions at all vertices. This forces Neumann boundary conditions at the degree one vertices that form the boundary of F_n .

Since $\phi_{n,m}$ maps F_n onto F_m we can by composition use $\phi_{n,m}$ to map functions over F_m to functions over F_n by the convention $\phi_{n,m}^* f = f \circ \phi_{n,m}$. The same definition is used for Φ_n^* as well.

Proposition 2.2. *For $m < n$, $\phi_{n,m}^* \text{Dom}(\Delta_m) \subset \text{Dom}(\Delta_n)$.*

Theorem 2.2 ([14]). *There exists a self-adjoint Laplacian Δ on L such that $\Delta \Phi_n^* f = \Phi_n^* \Delta_n f$ for all $f \in \text{Dom}(\Delta_n)$ and for all n with domain*

$$(2.3) \quad \text{Dom}(\Delta) = \overline{\bigcup_{n=0}^{\infty} \Phi_n^* \text{Dom}(\Delta_n)}.$$

Furthermore

$$(2.4) \quad \begin{aligned} \sigma(\Delta) = & \bigcup_{k=0}^{\infty} \{\pi^2 k^2\} \cup \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} \{(k+1/2)^2 \pi^2 d_n^2\} \cup \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{k^2 \pi^2 d_n^2\} \\ & \cup \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \{k^2 \pi^2 d_n^2\} \cup \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \frac{k^2 \pi^2 d_n^2}{4} \right\} \end{aligned}$$

And multiplicities

$$(2.5) \quad 1, \quad 2^n, \quad 2^{n-1}(j_n - 2)d_{n-1}, \quad 2^{n-1}(d_{n-1} - 1), \quad 2^{n-2}(d_{n-1} - 1)$$

The method of calculating the spectrum is based on the fact that an eigenfunction of Δ_n can be localized between wormholes since interior wormholes are degree four vertices and it is possible for the function to be constant on two of the incoming edges and non-constant on the other two and still satisfy the Kirchoff matching conditions that all incoming first derivatives sum to zero. The spectrum is then determining by breaking down the graphs, F_n into subgraphs on which such eigenfunctions are supported and the multiplicities counted by counting the number of each of these subgraphs. These counting arguments will be revisited later in this paper.

3. SPECTRAL ZETA FUNCTIONS

Definition 3.1. Let Δ be a self-adjoint positive-definite operator with a discrete spectrum λ_i and multiplicities g_i . Then the spectral zeta function is defined, where convergent as

$$(3.1) \quad \zeta_{\Delta}(s) = \sum_{i=1}^{\infty} \frac{g_i}{\lambda_i^s}.$$

We will also denote the analytic continuation of this function as $\zeta_{\Delta}(s)$.

If one considers the Laplacian on $[0, 1]$ with Dirichlet boundary conditions the spectrum is $\{k^2\pi^2\}_{k=1}^{\infty}$ and all of multiplicity one. Then

$$(3.2) \quad \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{2s}} = \frac{1}{\pi^{2s}} \zeta_R(2s)$$

Where $\zeta_R(s)$ is the Riemann zeta function which is known to have a meromorphic continuation to the whole complex plane. In the case of a Laakso space with data $j_i = 2$

$$(3.3) \quad \zeta_L(s) = \frac{\zeta_R(2s)}{\pi^{2s}} \left(\frac{4(2^{2s-1} + 1)}{4^s(4^2 - 4)} + \frac{6(2^{2s-1} - 1)}{4^s(4^s - 2)} + \frac{2^{s+1} - 2 + 2^{2s}}{4^s} \right)$$

Which then also has a meromorphic continuation to the complex plane with poles at known locations. In [9] formulae for the spectral zeta functions of any Laakso space with periodic data $\{j_i\}$ are given as a rational complex valued function times the Riemann zeta function. A feature of the spectral zeta functions on Laakso spaces that does not appear in the interval case is the existence of poles for $\zeta_L(s)$ off of the real axis. These are referred to as complex dimensions [12]. In [4] the residues of $\zeta_L(s)$ are used to calculate the leading terms of the Weyl asymptotics for Δ which for Laakso spaces with periodic j_i have a log-periodic oscillating term of leading order.

Theorem 3.1. Suppose that F_{∞} is constructed in such a way that $F_0 \subset \mathbb{R}^d$ is compact with non-empty interior and $\phi_0(B_i)$ induces a self-similar cell structure on F_0 . Further assume that the B_i have empty interior. Then the associated spectral zeta function will have a tower of simple poles above the spectral dimension. Furthermore, if the spectral zeta function over F_0 is meromorphic on the entire complex plane so is the spectral zeta function over F_{∞} .

Proof. As shown in [18] the projective limit construction in this case will be a non-negative real and discrete spectrum. Because it is assumed that F_1 can be realized as an assembly of identical pieces that are scaled copies of F_0 that overlap only on B_1 which has empty interior then $\text{Dom}(\Delta_1) = \phi_1^*(\text{Dom}(\Delta_0)) \oplus \mathfrak{F}_1$ where \mathfrak{F}_1 are the eigenfunctions that are orthogonal to $\phi_i^* \text{Dom}(\Delta_0)$ in $L^2(F_1)$. By a geometrical argument these eigenfunction are piece-wise defined as eigenfunctions on scaled copied of F_0 with suitable matching conditions to assure the orthogonality. See [14] for the case of Laakso spaces. Thus $\sigma(\Delta_i|_{\mathfrak{F}_1}) = c_1 \sigma(\Delta_0)$ for some c_1 . By the self-similarity of the cell structure the constant c_1 is the same for all n not just $n = 1$. This gives rise to a geometric series over n whose summation has a series of simple poles over the spectral dimension of L . Such series will be the topic of Section 5. Another series of eigenfunctions could occur due to Neumann boundary

conditions but these will also have the same scaling and will merely provide another tower of simple poles.

The spectral zeta function over F_∞ is the sum over n of scaled copied of the spectral zeta function over F_0 plus a finite number of bootstrap terms. Because this is actually the same geometric summation as in the previous paragraph the sum is a meromorphic function after regularization if and only if each term is meromorphic. \square

4. CASIMIR EFFECT

The Casimir effect arising between conductors and the quantum vacuum can be viewed as a consequence of vacuum zero-point energy. Simply put, displacing conductors generates new boundary conditions for the quantized vacuum, which in turn alters the zero-point energy and gives rise to a negative energy gradient. It is experimentally verified [15] that two parallel uncharged conducting plates experience an attractive pressure given by

$$|P_C| = \frac{\pi^2 c \hbar}{240 d^4}.$$

That such an attraction has its origins in relativistic quantum mechanics is reflected by the appearance of both Planck's quantum mechanical \hbar and the relativistic c . However, the direction of the Casimir pressure generally depends on the geometry of the conductors with which one is working. While in the case of plates and cylinder the attraction is positive, both spherical shells and Laakso spaces exhibit repulsion [5].

In computing the the Casimir effect on Laakso spaces, we take the vacuum expectation of a self-adjoint Hamiltonian operator which represents the quantized electromagnetic field and whose spectrum yields the permissible energies for the system. In particular, as our boundary conditions depend on on some displacement parameter d , we obtain

$$E_{vac}(d) = \langle 0 | H(d) | 0 \rangle \propto \sum_{\lambda \in \sigma(\Delta)} \omega_\lambda(d) \propto \sum_{\lambda \in \sigma(\Delta)} \sqrt{\lambda(d)} \propto \zeta_{L(d)}(-1/2).$$

The Casimir pressure will therefore be proportional to a derivative of the Laakso spectral zeta function evaluated at $-1/2$. In [9], the authors looked at the Casimir effect on $j_n = j$ Laakso spaces that arose from conducting plates attached at nodes in the F_1 graph approximation and placed symmetrically about the center. At each point of intersection with L , the conducting plates imposed Dirichlet boundary conditions on eigenfunctions and Kirchoff boundary conditions were maintained at the other nodes. Following the above outline, we then computed a modified spectral zeta function subject to these new boundary conditions. To make sense of the energy gradient, we allowed the plates to move symmetrically from their original locations compressing and stretching the underlying space in a natural way. Moving plates closer together compressed the interior space and stretched the exterior. Conversely, moving the plates away from each other stretched the interior space and compressed the exterior.

5. FINITE APPROXIMATIONS TO LAAKSO SPACES

In this section we consider the Casimir effect on F_m with Laplacian Δ_m in the case of two perfectly conducting plates placed at opposite ends of the unit interval. These boundary conditions are simply the Dirichlet boundary conditions. By truncating the counting arguments mentioned in Section 2 we see that the spectrum of Δ_m is given by

$$(5.1) \quad \begin{aligned} \sigma(\Delta_m) = & \bigcup_{k=1}^{\infty} \{\pi^2 k^2\} \cup \bigcup_{n=1}^m \bigcup_{k=0}^{\infty} \{(k+1/2)^2 \pi^2 d_n^2\} \cup \bigcup_{n=1}^m \bigcup_{k=1}^{\infty} \{k^2 \pi^2 d_n^2\} \\ & \cup \bigcup_{n=2}^m \bigcup_{k=1}^{\infty} \{k^2 \pi^2 d_n^2\} \cup \bigcup_{n=2}^m \bigcup_{k=1}^{\infty} \left\{ \frac{k^2 \pi^2 d_n^2}{4} \right\} \end{aligned}$$

With multiplicities

$$(5.2) \quad 1, \quad 2^n, \quad 2^{n-1}(j_n - 2)d_{n-1}, \quad 2^{n-1}(d_{n-1} - 1), \quad 2^{n-2}(d_{n-1} - 1)$$

From which the spectral zeta function for Δ_m , denoted $\zeta_m(s)$, can be easily calculated in the case where $j_i = j$ and $m \geq 4$ to be

$$(5.3) \quad \begin{aligned} \zeta_m(s) = & \frac{\zeta_R(2s)}{\pi^{2s}} \left[1 + \frac{1 - (2j^{-2s})^m}{j^{2s}(1 - 2j^{-2s})} \left(2^{1+2s} - 3 - \frac{2}{1 - 2j^{-2s}} \right) \right. \\ & \left. + \frac{1 - (2j^{1-2s})^m}{j^{2s}(1 - 2j^{1-2s})} \left(j + \frac{4^{2s+2} j^{1-2s}}{j^{2s}(1 - 2j^{1-2s})} \right) \right] \end{aligned}$$

The observed poles arise from the use of the summation formula for geometric series

$$(5.4) \quad \sum_{n=1}^m r^n = r \frac{1 - r^m}{1 - r},$$

which are not removable due to the interactions between the several sums being taken simultaneously. Since the poles found are outside of the domain of convergence for the summation over k these poles can only be approached through an analytic continuation and so we can choose to represent the summations using this formula since the extended functions will agree on an open subdomain and hence everywhere.

Proposition 5.1. *For all T -periodic sequences $\{j_i\}$ the spectral zeta function corresponding to Δ_n on F_n for $n \geq 3T$ have towers of complex poles.*

Proof. It has already been seen for constant sequences j_i . For periodic sequences with longer periods, the summation methods are augmented by summing over an individual period then summing over all periods. The requirement that n is large enough for three periods is so that the geometric aspect of the summation over n is fully present. \square

6. CASIMIR EFFECT ON L

This is the original setting in which the authors considered a Casimir effect in [9]. In addition to the boundary of the Laakso space playing the part of one pair of plates with Neumann boundary conditions we insert another symmetrically placed pair of plates in the interior so that the dependence of Casimir force on the separation of the plates can be explored. In this section we will consider for the sake

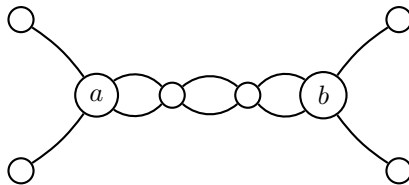


FIGURE 1. The F_1 graph for $j = 5$ where the interior plates are placed at nodes a and b . Here $X_0 = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$ and $Z = 3$.

of simplicity Laakso spaces with constant sequences $j_i = j$. The interior plates will be places symmetrically at level one wormhole locations. That is their location in the unit interval will be taken from L_1 . Set X_0 to be one half the distance between the interior plates, this gives the distance of the plates from the “center” of the Laakso space. Let Z be the number of nodes between the plates in the F_1 graph approximation of L . See Figure 1 for an example.

Definition 6.1. *Given a Laakso space with $j_i = j$ and a symmetrically located pair of plates whose location is determined by j and a chosen Z . The operator Δ' on $L^2(L, \mu)$ acts as Δ but with domain determined by imposing Dirichlet boundary conditions at the interior plates.*

Theorem 6.1 ([9] Theorem 4.2). *The operator Δ' is self-adjoint and has spectrum with multiplicities as given in Figure 2*

Using this it is a tedious but straightforward task to calculate $\zeta_{\Delta'}(s)$. Since we will be interested in how $\zeta_{\Delta'}(s)$ varies as j , Z , and X_0 are varied we will make the dependence explicit by writing $\zeta_{\Delta'}(s) = \zeta_{j, X_0, Z}(s)$.

Corollary 6.1. *Given a Laakso space with $j_i = j$, and conducting plates placed according to X_0 and Z we have the spectral zeta function given in Figure 3.*

Proposition 6.1. *The Casimir energy of a Laakso space given by $j_i = j$ and plates positioned according to X_0 and Z is proportional to $\zeta_{j, X_0, Z}(-1/2)$ and the self-exerted Casimir force due to this energy is proportional to Figure 5.*

Proof. See [9] for details. □

To see how Casimir force can vary with Z for a given j see Figure 4. Inspecting the expression in Figure 5 it is readily apparent that the force depends on the plate separation X_0 as a^{-2} instead of the expected a^{-d_s-1} . The parameter Z represents how many cells separate the plates, a sort of geometric distance. The dependence on Z is easily seem to be quadratic in Figure 5. Recall that Laakso spaces all have walk dimension $d_w = 2$ so $d_s = d_h = 1 + \frac{\log(r)}{\log(2)}$. The reason this happens is that Δ and consequently Δ' are truly one-dimensional operators rather than d_s -dimensional operators so the rapid grown in the eigenvalue counting function is due to the geometry of Laakso spaces rather than dimensionality of the Laplacian.

7. A HIGHER DIMENSIONAL CASE

7.1. The 3+ dimensional model. Let L_j be the Laakso space represented by the sequence $j_n = j \forall n$, let K be the Cantor set, and let $I = [0, 1]$. We modify the configuration in [9] by considering the space

$$\begin{aligned}
\sigma(\Delta') = & \bigcup_{k=1}^{\infty} \left\{ \left[\frac{k\pi}{2X_0} \right]^2 \right\} \cup \bigcup_{k=0}^{\infty} \left\{ \left[\frac{(k+1/2)\pi}{(1-2X_0)/2} \right]^2 \right\} \\
& \cup \bigcup_{k=0}^{\infty} \left\{ \left[(k+1/2)\pi \frac{j-(Z+1)}{1-2X_0} \right]^2 \right\} \cup \bigcup_{k=1}^{\infty} \left\{ \left[k\pi \frac{j-(Z+1)}{1-2X_0} \right]^2 \right\} \\
& \cup \bigcup_{k=1}^{\infty} \left\{ \left[k\pi \frac{Z+1}{2X_0} \right]^2 \right\} \cup \bigcup_{n=2}^{\infty} \bigcup_{k=0}^{\infty} \left\{ \left[I_n(k+1/2)\pi \frac{1-\frac{Z+1}{j}}{1-2X_0} \right]^2 \right\} \\
& \cup \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \left[I_n k\pi \frac{1-\frac{Z+1}{j}}{1-2X_0} \right]^2 \right\} \cup \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \left[I_n k\pi \frac{1-\frac{Z+1}{j}}{2(1-2X_0)} \right]^2 \right\} \\
& \cup \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \left[I_n k\pi \frac{Z+1}{2jX_0} \right]^2 \right\} \cup \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \left[I_n k\pi \frac{Z+1}{4jX_0} \right]^2 \right\}.
\end{aligned}$$

and multiplicities are listed in the same order

- 1) 1;
- 2) 2;
- 3) 2;
- 4) $j - Z - 3$;
- 5) $Z + 1$;
- 6) 2^n ;
- 7) $(1 - \frac{Z+1}{j})I_{n-1}2^{n-1}(j-2) + 2^{n-1}(1 - \frac{Z+1}{j})I_{n-1}$;
- 8) $2^{n-2}[(1 - (\frac{Z+1}{j})I_{n-1} - 1) - 2^{n-2}]$;
- 9) $\frac{Z+1}{j}I_{n-1}2^{n-1}(j-2) + 2^{n-1}\frac{Z+1}{j}I_{n-1} + 2^{n-1}$;
- 10) $2^{n-2}[\frac{Z+1}{j}I_{n-1} - 1]$.

FIGURE 2.

$$L_j \times \mathbb{R}^2 = [(I \times K)/\sim] \times \mathbb{R}^2.$$

Attach two conducting plates $P_1, P_2 \subset L \times \mathbb{R}^2$ where

$$P_1 = [(0 \times K)/\sim] \times \mathbb{R}^2; P_2 = [(1 \times K)/\sim] \times \mathbb{R}^2.$$

Now allow symmetric displacement of the plates in the I direction of the Laakso space. By construction, plates moving towards one another will compress the interior space between them while plates moving apart will stretch it.

7.2. Casimir Effect Strength versus Plate Separation Distance. The self-adjoint operator $\Delta_{L_j \times \mathbb{R}^2}$ takes the form

$$\Delta_{L_j \times \mathbb{R}^2} = \Delta_{L_j} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$$

where Δ_{L_j} is the non-negative definite self-adjoint Laplacian on L_j such that $\Delta \Phi_n^* f = \Phi_n^* \Delta_n f$ for all $f \in \text{Dom}(\Delta_n)$. The generalized spectrum for $\Delta_{L_j \times \mathbb{R}^2}$ is conveniently described by

$$\begin{aligned}
\zeta_{j,X_0,Z}(s) = & \sum_{k=0}^{\infty} \frac{2}{[(2k+1)\pi/(1-2X_0)]^{2s}} + \sum_{k=1}^{\infty} \frac{1}{[k\pi/(2X_0)]^{2s}} \\
& + \sum_{k=1}^{\infty} \frac{(j-Z-3)}{[jk\pi \frac{(1-(Z+1)/j)}{1-2X_0}]^{2s}} + \sum_{k=1}^{\infty} \frac{Z+1}{[k\pi(Z+1)/(2X_0)]^{2s}} \\
& + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{2^n}{[I_n(k+1/2)\pi(1-\frac{Z+1}{j})/(1-2X_0)]^{2s}} \\
& + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{(1-\frac{Z+1}{j})2^{n-1}I_{n-1}(j-2) + 2^{n-1}(1-(Z+1)/j)I_{n-1}}{[I_n k\pi \frac{(1-(Z+1)/j)}{(1-2X_0)}]^{2s}} \\
& + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{2^{n-2}[(1-(Z+1)/j)I_{n-1}-1] - 2^{n-2}}{[I_n k\pi(1-(Z+1)/j)/[2(1-2X_0)]]^{2s}} \\
& + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{(Z+1)/j[2^{n-1}I_{n-1}(j-2) + 2^{n-1}I_{n-1}] + 2^{n-1}}{[k\pi I_n(Z+1)/(2jX_0)]^{2s}} \\
& + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \frac{(Z+1)/j[2^{n-2}I_{n-1}] - 2^{n-2}}{[I_n k\pi/(4jX_0)]^{2s}}.
\end{aligned}$$

FIGURE 3.

$$\sigma_{L_j \times \mathbb{R}^2} = \{\lambda_s + k_{x_1}^2 + k_{x_2}^2 : \lambda_s \in \sigma_{L_j}\}.$$

As mentioned in Section 4, the zero-point energy for our system arises from the ground state expectation of the Hamiltonian representing the quantized electric field, which takes the form

$$(7.1) \quad E_{Cas} = 2 \cdot \frac{\hbar}{2} \sum_{\lambda_s, k_{x_1}, k_{x_2}} \omega_{\lambda_s, k_{x_1}, k_{x_2}} = \hbar c \sum_{\lambda_s, k_{x_1}, k_{x_2}} \sqrt{\lambda_s + k_{x_1}^2 + k_{x_2}^2}.$$

The quantity $\hbar\omega_{\lambda_s, k_{x_1}, k_{x_2}} = \hbar c \sqrt{\lambda_s + k_{x_1}^2 + k_{x_2}^2}$ is the electromagnetic energy associated to radiation with wave vector $(\sqrt{\lambda_s}, k_{x_1}, k_{x_2})$.

We recast this last expression as an integral and switch to polar coordinates:

$$\begin{aligned}
\frac{E_{Cas}}{Area} &= \frac{\hbar c}{4\pi^2} \sum_{\lambda_s \in \sigma_{L_j}} \int_{\mathbb{R}^2} dk_{x_1} dk_{x_2} \sqrt{\lambda_s + k_{x_1}^2 + k_{x_2}^2} \\
&= \frac{\hbar c}{4\pi^2} \sum_{\lambda_s \in \sigma_{L_j}} \int_0^\infty \int_0^{2\pi} dk \cdot k \sqrt{k^2 + \lambda_s} d\theta dk.
\end{aligned}$$

We regularize this last quantity to obtain

$$\frac{E_{Cas}}{Area} = \frac{\hbar c}{6\pi} \sum_{\lambda_s \in \sigma_{L_j}} (\lambda_s)^{3/2}.$$

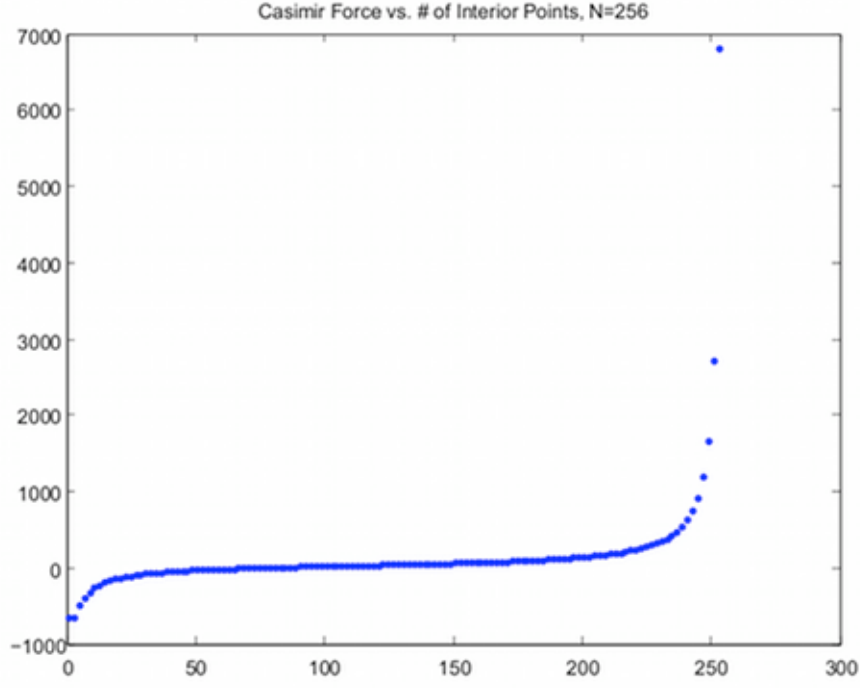


FIGURE 4. Values of the Casimir force plotted for $j = 256$ and Z ranging between 1 and 125. Notice that for large Z we see an interaction between the plates and the boundary of the Laakso space while for small Z see a similar interaction between the two plates.

Proposition 7.1. *Two conducting plates attached at the boundary of $L_j \times \mathbb{R}^2$ as described in the set up will experience a pressure given by*

$$\begin{aligned}
 P_{Cas}(j) &= \frac{\hbar c \pi^2}{240} \left[1 + \frac{2j^4}{1-2j^3} + \frac{17}{8} \cdot \left(\frac{j^7}{1-2j^4} - \frac{j^6}{1-2j^3} \right) + \right. \\
 &\quad \left. \left(\frac{j^4}{1-2j^4} - \frac{2j^3}{1-2j^4} \right) \right] \\
 &= \frac{\hbar c \pi^2}{240} \left[\frac{8 - 16j^3 - 8j^4 + 15j^6 + j^7}{8 - 16j^3 - 16j^4 + 32j^7} \right],
 \end{aligned}$$

where a positive signed pressure indicates a repulsive force.

Proof. Computing the Casimir pressure involves eigenvalue counting arguments similar to the ones made in [14] except that Neumann boundary conditions are replaced by Dirichlet boundary conditions at the boundary $(0 \times K)/\sim$ and $(1 \times K)/\sim$. We construct the full spectrum of Δ_{L_j} by exploiting orthogonality relations between eigenfunctions in different quantum graph approximations. Decomposing each quantum graph F_n into loops, V's, and crosses, note that the V's are the only shapes whose localized eigenfunctions are altered by the new boundary conditions.

$$\begin{aligned}
F_C(j, Z) &\propto \frac{d}{dx} \zeta_{j,x,Z} \left(-\frac{1}{2} \right) \Big|_{x=X_0} \\
&= \frac{(j - (Z + 1))}{24(1 - 2j)(1 - 2X_0)^2} - \frac{(j - (Z + 3))(j - (Z + 1))}{12(1 - 2X_0)^2} \\
&\quad - \frac{j^3(j - 2)(1 - \frac{Z+1}{j})^2}{12(1 - 2j^2)(1 - 2X_0)^2} - \frac{(1 - \frac{Z+1}{j})j^2(j - (Z + 1))}{24(1 - 2X_0)^2(1 - 2j^2)} \\
&\quad + \frac{(1 - \frac{Z+1}{j})j^2}{24(1 - 2X_0)^2(1 - 2j)} + \frac{(Z + 1)^2}{48X_0^2} + \frac{(Z + 1)^2(j - 2)}{24X_0^2(1 - 2j^2)} \\
&\quad + \frac{j(Z + 1)^2}{96(1 - 2j^2)X_0^2} + \frac{1}{6(1 - 2X_0)^2} - \frac{j^2(1 - \frac{Z+1}{j})}{24(1 - 2j)(1 - 2X_0)^2} \\
&\quad + \frac{j(Z + 1)}{96X_0^2(1 - 2j)} - \frac{j^2(1 - \frac{Z+1}{j})}{12(1 - 2X_0)^2(1 - 2j)} + \frac{1}{48X_0^2} + \frac{(Z + 1)j}{48X_0^2(1 - 2j)}
\end{aligned}$$

FIGURE 5.

Once the new spectrum with multiplicities is found, substitute into the expression for Casimir energy, regularize the sum, and take derivatives with respect to displacement to obtain the result. \square

In the limit $j \rightarrow \infty$ the Hausdorff dimension of the Laakso spaces L_j go to 1. So in the $j \rightarrow \infty$ limit we might expect something similar to the classical case since the localized eigenfunctions have eigenvalues increasing without bound so in the limit effectively vanish. Simply from the formula we have that $\lim_{j \rightarrow \infty} P_{Cas}(j) = \frac{1}{32} \frac{\hbar c \pi^2}{240}$. Hence, in the limit, our fractal pressure is precisely 1/32 the magnitude of the classical Casimir value and acting in the opposite direction.

Corollary 7.1. *Two conducting plates attached at the boundary of $L_j \times \mathbb{R}^2$ and then stretched to a distance d from one another will experience a Casimir pressure*

$$P_{Cas}(j, d) = \frac{P_{Cas}(j)}{d^4}.$$

Proof. This follows immediately from the fact that multiplying the displacement by a factor of d means $\lambda_s \rightarrow \frac{\lambda_s}{d^2}$ for every $\lambda_s \in \sigma_L$. \square

In particular, the power law governing the Casimir pressure as a function of displacement is independent of the spectral dimension of the Laakso space L_j .

7.3. Casimir Pressure as a function of $\{j_i\}$.

Proposition 7.2. *Let L is the Laakso space represented by some N -periodic sequence $\{j_i\}$. Then two conducting plates attached at the boundary of $L \times \mathbb{R}^2$ yield an unnormalized Casimir energy given by*

$$\begin{aligned}
E_{Cas}(\{j_i\}, d) &= \sum_{k=1}^{\infty} \left(\frac{k\pi}{d} \right)^3 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{k\pi I_n}{d} \right)^3 \\
&\quad + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} 2^{n-1} I_{n-1} (j_n - 2) \left(\frac{k\pi I_n}{d} \right)^3 \\
&\quad + \frac{17}{16} \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} 2^{n-1} (I_{n-1} - 1) \left(\frac{k\pi I_n}{d} \right)^3.
\end{aligned}$$

Proof. This more general case follows from Equation 7.1 and eigenvalue counting arguments similar to the ones made in [14] for periodic Laakso spaces. \square

Proposition 7.3. *Let L be the Laakso space represented by some N -periodic sequence $\{j_i\}$. Then two conducting plates attached at the boundary of $L \times \mathbb{R}^2$ will experience a Casimir pressure given by*

$$\begin{aligned}
P_{Cas}(\{j_i\}, d) &= \frac{\hbar c \pi^2}{240 d^4} \left[1 + \frac{15}{32} \left(\sum_{i=1}^N \prod_{k \leq i} 2j_k^3 \right) \left(\frac{1}{1 - r^{3N} 2^N} \right) \right. \\
&\quad + \frac{1}{2} \left(\sum_{i=1}^N 2j_k^4 \right) \left(\frac{1}{1 - r^{4N} 2^N} \right) \\
&\quad \left. - \frac{15}{32} \sum_{i=1}^N \left(\prod_{k \leq i} 2j_k^4 \right) \left(\frac{j_i^{N-i}}{j_i^N - r^{4N} 2^N} \right) \right]
\end{aligned}$$

where a positive sign indicates a repulsive force.

Proof. Note that one formally has

$$\sum_{n=1}^{\infty} I_n = \left(\sum_{l=0}^{\infty} r^{Nl} \right) \left(\sum_{i=1}^N \prod_{k \leq i} j_k \right).$$

Hence, we can regularize

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} 2^n \left(\frac{k\pi I_n}{d} \right)^3 &= \frac{\zeta(-3)\pi^3}{d^3} \sum_{n=1}^{\infty} 2^n I_n^3 \\
&= \frac{\zeta(-3)\pi^3}{d^3} \sum_{l=0}^{\infty} r^{3Nl} \left(\sum_{i=1}^N 2^{Nl+i} \prod_{k \leq i} j_k^3 \right) \\
&= \frac{\zeta(-3)\pi^3}{d^3} \left(\sum_{l=0}^{\infty} r^{3Nl} 2^{Nl} \right) \left(\sum_{i=1}^N \prod_{k \leq i} 2j_k^3 \right) \\
&= \frac{\pi^3}{120 d^3} \left(\frac{1}{1 - r^{3N} 2^N} \right) \left(\sum_{i=1}^N \prod_{k \leq i} 2j_k^3 \right).
\end{aligned}$$

The unregularized Casimir energy for $L \times \mathbb{R}^2$ is comprised of terms that look quite similar to above sum. The reader may obtain the formula for Casimir pressure

by first regularizing each term of the Casimir energy in Proposition 7.2 and then taking a (negative) spatial derivative with respect to d . \square

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